Some comments on bound state eigenvalues of PT-symmetric potentials

S. Banerjee & R. Roychoudhury*
Physics & Applied Mathematics Unit
Indian Statistical Institute
203 B. T. Road
Kolkata 700035
INDIA

ABSTRACT

Using purely physical arguments, it is claimed that for ID Schrödinger operators with complex PT-symmetric potentials having a purely real, attractive potential well and a purely imaginary repulsive part, bound state eigenvalues will be discrete and real. This has been illustrated with several potentials possessing similar properties.

***E-mail : raj @ www.isical.ac.in**

Complex potentials are encountered in a variety of diverse situations ranging from conventional quantum mechanical scattering[1] problems to field theory, population biology and quantum chemistry[2]. The solution of Schrödinger's equation with such complex potentials is complicated by the fact that the hermiticity of the Hamiltonian is lost because it is not invariant to parity (P) or time reversal (T). Inspired by the result obtained by Bessis, Bender et al[3] suggested that the commutation of the product PT is a possible mechanism of weakening the standard requirements of hermiticity, and the so-called "PT-symmetric" quantum mechanics acquired revewed interest[4-14].

For such Hamiltonians, the Schrödinger operator

$$H = \frac{d^2}{dx^2} + V(x) \tag{1}$$

is PT invariant if the potential $V(x)$ satisfies

$$[V(x)] = [V(-x)]^* \tag{2}$$

We consider a class of such complex potentials which can be represented in the so called supersymmetric form [13].

$$V^{(1),(2)} = u^2 \pm u' \tag{3}$$

where u is a complex function of x and prime denotes differentiation $w.\ r.\ t.\ x.$ u can be expressed explicitly as $a(x) + ib(x)$ where $a(x)$ and $b(x)$ are certain real, continuously differentiable functions in R. So we have

$$V^{(1),(2)} = \left(a^2 - b^2 \pm a'\right) + 2iab \pm b' \tag{4}$$

In earlier literature involving exactly solvable as well as numerical and WKB[14] procedure based eigenvalues, one important feature of many complex potentials has not been mentioned. Recently Ahmed[21] observed that for certain asymptotically vanishing potentials all the eigen values are real when the real part is stronger than the imaginary part.We wish to comment, sololy on the basis of physical arguments, that, an inspection of these potentials reveal that only real, bound state, discrete eigenvalues can be present due to the repulsive nature of the imaginary part of the complex potential and presence of a purely real potential well. We elaborate this point with a few illustrative examples of the type given by Eqn.(3) and our arguments are valid for harmonic and cubic as well as complex anharmonic oscillators. For the sake of simplicity coupling constants are taken to be unity.

For complex potentials $V(x) = V_R(x) - iV_I(x)$ where V_R, V_I are real, the differential conservation relation for the position probability density[15] is given by

$P(x,t) = u^*(x,t)\ u(x,t) = |u(x,t)|^2$, (u being the wave function) and vector probability current density

$$\begin{aligned} S(x,t) &\sim [u^* \nabla u - (\nabla u^*) u] \\ \frac{\partial P}{\partial t}\ (x,t) + \nabla \cdot S(x,t) &= -\frac{2V_I}{\hbar}\ P(x,t) \end{aligned} \tag{5}$$

since $P(x,t)$, is non-negative, $V_I \geq 0$ indicates that the R. H. S. of eqn.(5) is a sink, whereas $V_I < 0$ is a source of probability. Therefore, from this probability conservation constraint and for physical problems of interest, like inelastic neutron scattering and absorption from the nucleus, as discussed in[6], we assume that $V_I \geq 0$, i e the sign of imaginary part of the complex potential is dictated by these physical requirements.

The first example that came to illustrate these remarks is a localized potential belonging to the category described by eqn.[3]. Here the functions $a(x)$ and $b(x)$ are given by $a(x) = \dfrac{1}{x}$ and $b(x) = \dfrac{\lambda}{x^2}$, λ being a real coupling constant whose sign is determined by the considerations described in the previous paragraph. The supersymmetric potential pair is constructed as follows : Define $W^+ = a + ib$ and $W^- = a - ib$. Then,

$$V^{(1)} \ = \ W^{+2} + W^{+\prime} = -\frac{\lambda^2}{x^4} \tag{6a}$$

and

$$V^{(2)} \ = \ W^{-2} - W^{-\prime} = \frac{2}{x^2} - \frac{\lambda^2}{x^4} + \frac{4i\lambda}{x^3} \tag{6b}$$

where λ is positive, from preceding arguments. This pair of potentials of eqn.(6) vanish at infinity faster than coulombic type potentials ($\frac{1}{x}$) and are plotted in figs. $1(a-c)$ with figs. 1(a) and (c) depicting the real parts of $V^{(1)}$ and $V^{(2)}$ and fig. 1(b) the imaginary part of $V^{(2)}$. For this supersymmetric pair, the real parts start out at $-\infty$ and constitute an attractive potential well below the real axis before becoming vanishingly small at large distances from the origin. On the other hand, for the imaginary part of the complex potential $V^{(2)}$, (which is ≥ 0, from the discussions of eqn.(5)), the potential starts at $+\infty$ at the origin and falls off to zero as the inverse cube of the distance from the origin.

This is a purely repulsive potential which cannot contribute to bound state eigenvalues. So, for this supersymmetric potential pair, the solution of Schrödinger's equation for bound state eigenvalues essentially constitutes the solution of eqn.(1) for real potentials, thus preserving hermiticily and real discrete bound state eigenvalues can be obtained in principle.

The second and third supersymmetric pair of potentials to be discussed are non localized and obey eqn (3). They are[16]

$$w_1 = \frac{1}{x+i} - i(x+i)^2 \tag{7a}$$

$$w_2 = -\left[\frac{1}{x-i} - i(x-i)^2\right] \tag{7b}$$

and [9]

$$V_1^-(x) = \frac{2}{(x+i)^2} - (x+i)^4 \tag{8a}$$

$$V_2^-(x) = -4i(x-i) - (x-i)^4 \tag{8b}$$

These two pairs of manifestly PT-symmetric potentials have been plotted in figs.2(a-d) and 3(a-b) respectively, the second pair in fig.3 viz 3 and 3d being almost identical is not plotted. Again, it found that the imaginary part of such potentials do not form attractive potential wells and so cannot contribute to the solution of Schrödinger's equation for the bound state problem, which reduces to a problem with a real potential well for the Schrödinger operator of eqn.(1).

Another PT-symmetric potential pair which obeys eqn.(3) is the supersymmetric version of the modified Poeschl-Teller hole with coupling constant μ (and parameters λ and $\tilde{\lambda}(\tilde{\lambda}-1) = \frac{\lambda^2}{\mu^2} - \frac{1}{4}$), which has also been solved analytically[21] elsewhere, and which we include here for argument's sake

$$V_P^{(1)} = \frac{\mu^2}{4} - \mu^2\left[\tilde{\lambda}(\tilde{\lambda}-1)+1\right]\ sech^2\ \mu x - 2i\lambda\mu\ sech\ \mu x\ tanh\ \mu x \tag{9a}$$

$$V_P^{(2)} = \frac{\mu^2}{4} - \mu^2\tilde{\lambda}(\tilde{\lambda}-1)\ sech^2\ \mu x \tag{9b}$$

The first of these potentials are plotted in fig.4(a) (for the real part) and fig.4(b) for the imaginary part. The real part constists of a bounded potential well with discrete eigenvalues[13] whereas, the imaginary part is a repulsive potential due to the conditions imposed by eqn.(5) (the coupling constants μ, λ having the appropriate signs) and does not contribute to the real discrete bound state eigenvalues.

For harmonic and anharmonic oscillator types of potentials with positive coupling constants for the real harmonic and biharmonic components, no bound state eigenvalues are produced although real, discrete eigenvalues extending to continium have been obtained. For these harmonic oscillators, negative coupling constants on the other hand would lead to discrete, bound states.

For a potential of the type given in [17] viz., a real harmonic oscillaor (with coupling constant $\mu \geq 0$) and a cubic oscillator with a purely imaginary coupling constant g viz., $V(x) = \mu x^2 + igx^3$, where g is necessarily positive as a consequence of eqn.(5), it is well known that the real part (fig.5(a)) gives real, discrete, equidistant eigenvalues which extend to infinity. The imaginary part, however, (fig.5(b)) does not possess any attractive potential well and therefore cannot contribute to bound state, real eigenvalues, either.

Finally, for a complex anharmonic oscillator potential[1] $V(x) = ax^4 + bx^3 + cx^3 + dx$, with $b = i\beta$ and $d = i\delta$ for PT-symmetry, it was earlier asserted that the coupling constant has to be negative[19] for bound states, but recent suggestions[14, 19] state that in spite of the manifest non-hermiticity of the related Hamiltonian, the procedure of quantization may be kept equally well defined at any sign of a. Thus the real part of this anharmonic oscillator can have (fig.6 (a)) discrete, real, bound states, whereas the imaginary part (fig.6 (b)) which is dominated by the $i\beta x^3$ term, is positive and does not have a potential well and so bound states are not possible so, in conclusion it may be stated that for non-hermition Hamiltonians, whose complex potentials obey PT-symmetry expressed by eqn.(2), physical arguments support the presence of real bound state eigenvalues obtained only from the real part of the potential, which constitutes an attractive potential well, whereas the purely repulsive imaginary part does not contribute to these bound states. This postulate is illustrated with several potentials that have been plotted with the magnitudes of the coupling constants taken to be unity, for simplicity and without loss of generality. However, some important exceptions[18, 20] have been found for which non-hermitian Hamiltonians $H \neq H^+$ support perfectly stable bound states. Recently, the presence of such exceptional stable resonances for which $Im\ E = 0$ (E eigenvalue) has been a subject of intensive study. To conclude we argue that PT-symmetric potentials having a purely real attractive potential well and a purely imaginary repuslive part will have discrete and real eigen values.

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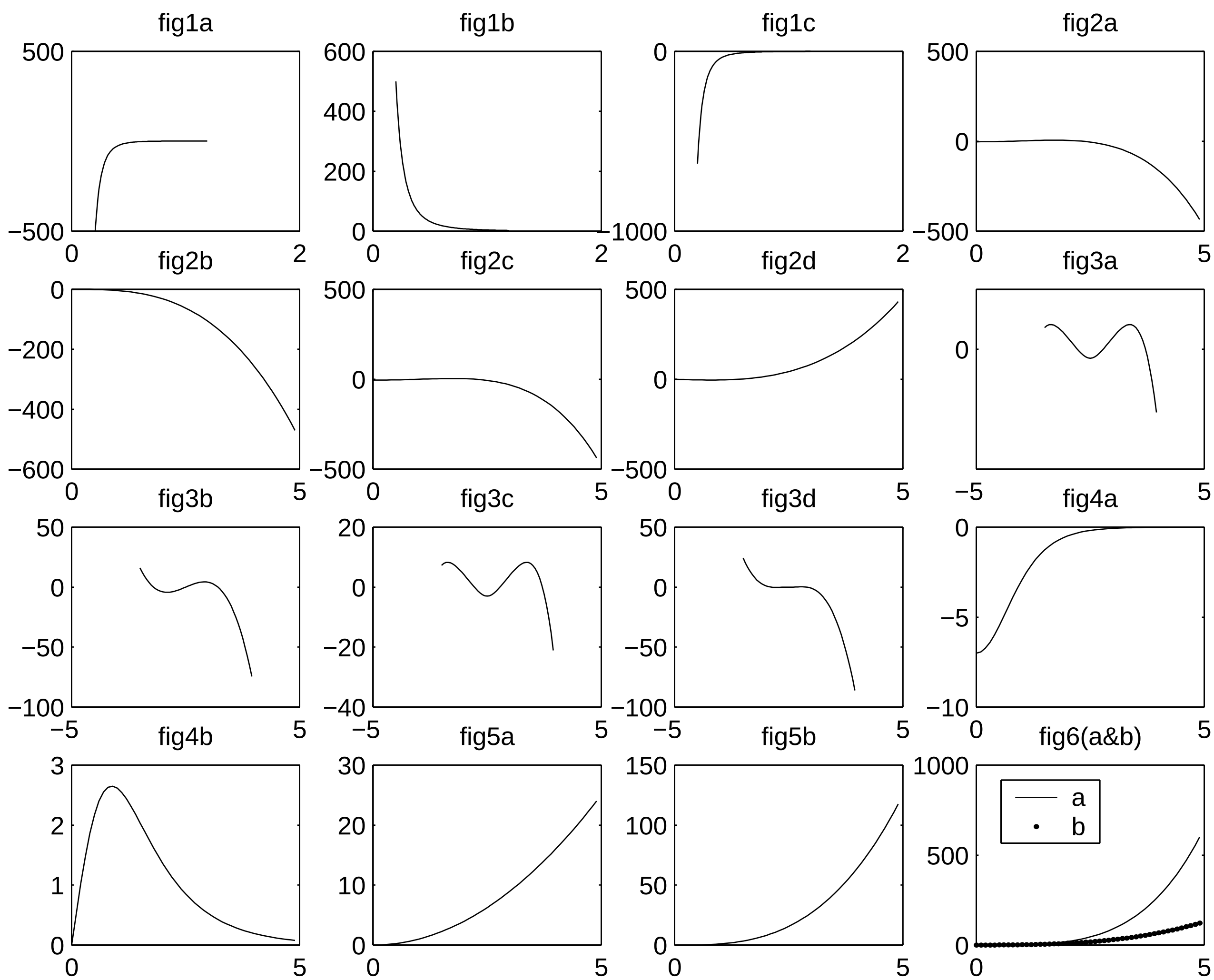
fig1a
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fig6(a&b)
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